

MATRIX ALGEBRA

1. INTRODUCTION

The finite element analysis usually generates a large number of simultaneous equations. Therefore it is impossible to solve these equations without use of an electronic computer. To ensure systematic and efficient solution from these computers, it is imperative to employ matrix method of analysis. Hence, the understanding of basic principles of matrix algebra is more essential to understand FEM.

The effectiveness of using matrices in practical calculations is readily realized by considering the following set of simultaneous equations.

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \text{-----} & a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \text{-----} & a_{2n}x_n & = & b_2 \\ \left| \begin{array}{c} | \\ | \\ | \end{array} \right. & \left| \begin{array}{c} | \\ | \\ | \end{array} \right. & & \left| \begin{array}{c} | \\ | \\ | \end{array} \right. \\ a_{n1}x_1 + a_{n2}x_2 + \text{-----} & a_{nn}x_n & = & b_n \end{array}$$

where, $x_1, x_2, \text{-----} x_n$ are unknowns.

Using matrix notation, the above set of equations can be written as

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \text{-----} & a_{n1} \\ a_{21} & a_{22} & \text{-----} & a_{n2} \\ \left| \begin{array}{c} | \\ | \\ | \end{array} \right. & \left| \begin{array}{c} | \\ | \\ | \end{array} \right. & & \left| \begin{array}{c} | \\ | \\ | \end{array} \right. \\ a_{n1} & a_{n2} & \text{-----} & a_{nn} \end{array} \right]_{(n \times n)} \left\{ \begin{array}{c} x_1 \\ x_2 \\ | \\ x_n \end{array} \right\}_{(n \times 1)} = \left\{ \begin{array}{c} b_1 \\ b_2 \\ | \\ b_n \end{array} \right\}_{(n \times 1)}$$

The above equation can also be written in short form as

$$[A]\{x\} = \{b\}$$

where, $[A]$ is a square matrix of dimension $(n \times n)$

$\{x\}$ and $\{b\}$ are vectors of dimension $(n \times 1)$.

Ex. 1. Write the following set of simultaneous equations in the form of matrix.

$$5x_1 - 4x_2 + x_3 = 0$$

$$-4x_1 + 6x_2 - 4x_3 + x_4 = 1$$

$$x_1 - 4x_2 + 6x_3 - 4x_4 = 0$$

$$x_2 - 4x_3 + 5x_4 = 0$$

The above equation can written in matrix form as

$$\left[\begin{array}{cccc} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{array} \right] \left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right\}$$

2. SPECIAL MATRICES

Whenever the elements of matrices obey a certain law, those matrices are called special matrices.

In the following some of the special matrices are defined.

1. *Square matrix*: The square matrix is one in which number of rows is equal to number of columns and having the dimension $(n \times n)$.

For example,

$$A = \left[\begin{array}{ccc} 1 & 4 & 6 \\ 2 & 3 & 7 \\ 4 & 5 & 8 \end{array} \right]_{(3 \times 3)}$$

2. **Symmetric matrix:** A symmetric matrix is a square matrix whose elements satisfy $a_{ij} = a_{ji}$ i.e. the elements located symmetrically w.r.t principal diagonal are equal.

For example, $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 6 & -2 \\ 0 & -2 & 8 \end{bmatrix}$ is a symmetric matrix.

Note that the symmetric matrix is a square matrix but not vice versa i.e. a square matrix need not to be a symmetric matrix. The above symmetric matrix can also be written as

$$A = \begin{bmatrix} 2 & 1 & 0 \\ & 6 & -2 \\ \text{Sym} & & 8 \end{bmatrix}$$

3. **Diagonal matrix:** Is a square matrix with non-zero elements only along principal diagonal.

For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{(3 \times 3)}$

4. **Identity matrix or unit matrix:** Is a diagonal matrix with 1's along principal diagonal and zero elements outside the principal diagonal.

For example, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{(3 \times 3)}$

5. **Banded matrix:** A banded matrix is one in which all of the non-zero elements located within a band and outside of the band all elements are zero as shown in Fig. 1(a).

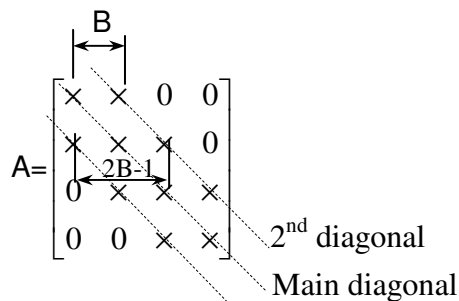


Fig. 1(a): Banded matrix

Where $(2B-1)$ is bandwidth and B is half bandwidth.

If banded matrix follows the relation $a_{ij} = a_{ji}$, then the matrix is called *symmetric banded matrix* and is written as shown in Fig. 1(b).

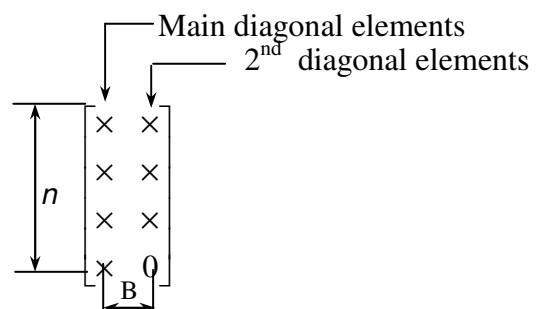
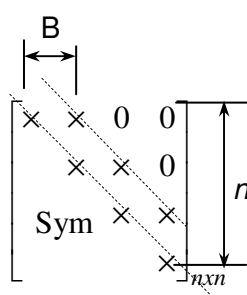


Fig. 1(b): Symmetric banded matrix

Fig. 1(c): Computer storage scheme

Advantage of symmetric banded matrix in FEM:

Fortunately, the stiffness matrices in finite element method are symmetric and banded matrices. The symmetric and banded stiffness matrices have the following advantages in core storage scheme of computers.

1. Since only the non-zero elements in banded matrix need to be stored, the elements of the matrix are compactly stored in the “ $n \times B$ ” matrix as shown in Fig. 1(c).
2. As the size of the matrix to be stored reduces from “ $n \times n$ ” matrix to “ $n \times B$ ” matrix a large sized matrix can be stored.
3. Another advantage of symmetric and banded matrix in FEM is, it is easy to apply Gauss elimination method while solving large number of simultaneous equations.

6. *Upper Triangular Matrix:* An upper triangular matrix is one whose elements below the principal diagonal are all zero.

For example. $\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ is a upper triangular matrix.

3. MATRIX OPERATIONS

1. *Matrix multiplication:* The product of “ $m \times n$ ” matrix A and “ $n \times p$ ” matrix B results in an “ $m \times p$ ” matrix C.

$$\text{i.e. } A_{(m \times n)} \bullet B_{(n \times p)} = C_{(m \times p)}$$

The $(i,j)^{th}$ component of C is obtained by taking the dot product.

$$\text{i.e. } C_{ij} = (i^{th} \text{ row of A}) \bullet (j^{th} \text{ column of B})$$

$$\text{For example: } A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 4 \\ 5 & -2 \\ 0 & 3 \end{bmatrix}$$

$$\text{Then, } C = AB = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 5 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2*1+1*5+3*0 & 2*4+1*(-2)+3*3 \\ 0*1+(-2)*5+1*0 & 0*4+(-2)*(-2)+1*3 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & 15 \\ -10 & 7 \end{bmatrix}$$

2. *Transpose of matrix:* The transpose of matrix $\mathbf{A}$, written as $\mathbf{A}^T$ is obtained by interchanging the rows and columns of $\mathbf{A}$. Thus the rows of $\mathbf{A}$ are the columns of $\mathbf{A}^T$.

$$\text{If } \mathbf{A} = \begin{bmatrix} 1 & -5 \\ 0 & 6 \\ -2 & 3 \\ 4 & 2 \end{bmatrix}_{4 \times 2}; \text{ Then } \mathbf{A}^T = \begin{bmatrix} 1 & 0 & -2 & 4 \\ -5 & 6 & 3 & 2 \end{bmatrix}_{2 \times 4}$$

In general, if $\mathbf{A}$ is of dimension “ $m \times n$ ”, then $\mathbf{A}^T$ is of dimension “ $n \times m$ ”.

Note: It is important to point out some special rules concerning to the transposed matrices and their use in matrix multiplications.

1. If $\mathbf{A}$ is a symmetric square matrix, then $\mathbf{A} = \mathbf{A}^T$.
2. If $\mathbf{C} = \mathbf{AB}$, then $\mathbf{C}^T = \mathbf{B}^T \mathbf{A}^T$.

3. If $D = ABC$, then $D^T = C^T B^T A^T$. If B is symmetric square matrix, $D = C^T B A^T$.

3. **Determinant of a matrix:** The determinant of a square matrix A is a scalar quantity denoted as " $\det A$ " or " $|A|$ ". The determinant of " $n \times n$ " matrix A can be defined in terms of determinant of sub-matrices of A and is obtained by the *Recurrence formula* as given below.

$$\det A = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det M_{1j}$$

where minor M_{1j} is the $(n-1) \times (n-1)$ matrix obtained by eliminating 1st row and j^{th} column of the given matrix A .

For example, the determinant of (2×2) and (3×3) matrices given below are obtained as

$$(a) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad (b) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution:

(a) Here $n = 2, j = 1, 2$

$$\det A = \sum_{j=1}^2 (-1)^{1+j} a_{1j} \det M_{1j} = (-1)^{1+1} a_{11} \det M_{11} + (-1)^{1+2} a_{12} \det M_{12} \quad \text{---(a)}$$

where, M_{11} is obtained by eliminating 1st row and 1st column of A and M_{12} is obtained by eliminating 1st row and 2nd column of A .

Therefore, $M_{11} = a_{22}$ and $M_{12} = a_{21}$

And $\det M_{11} = a_{22}$ and $\det M_{12} = a_{21}$. Substitute into (a).

$$\det A = (-1)^2 a_{11} a_{22} + (-1)^3 a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21}$$

(b) Here $n=3, j=1,2,3$

$$\det A = (-1)^{1+1} a_{11} \det M_{11} + (-1)^{1+2} a_{12} \det M_{12} + (-1)^{1+3} a_{13} \det M_{13} \quad \text{---(a)}$$

where,

$$\det M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} a_{33} - a_{23} a_{32}$$

$$\det M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21} a_{33} - a_{23} a_{31} \quad \text{and} \quad \det M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21} a_{32} - a_{22} a_{31}$$

Substituting into Eqn. (a), we get

$$\begin{aligned} \det A &= (-1)^2 a_{11} (a_{22} a_{33} - a_{23} a_{32}) + (-1)^3 a_{12} (a_{21} a_{33} - a_{23} a_{31}) + (-1)^4 a_{13} (a_{21} a_{32} - a_{22} a_{31}) \\ &= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} \end{aligned}$$

Prob. (1). Evaluate the determinant of matrix A, if $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. (Ans. 8).

4. **Matrix inversion:** The inversion of a matrix A is written as A^{-1} . If $\det A \neq 0$, then A has an inverse. The matrix that possesses an inverse is said to be non-singular. On the other hand, the matrix without an inverse (i.e. $\det A = 0$) is said to be singular matrix.

The elements of A^{-1} satisfy the relation $A^{-1} A = A A^{-1} = I$

Inverse of a square matrix A is obtained by the relation,

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{C^T}{\det A}$$

where, $\text{adj } A = C^T$. The matrix C with elements C_{ij} in the above equation is called *cofactor matrix* and is obtained by

$$C_{ij} = (-1)^{i+j} \det M_{ij}$$

where, minor M_{ij} of a square matrix $\mathbf{A}$ is the $(n-1 \times n-1)$ matrix obtained by eliminating i^{th} row and j^{th} column of $\mathbf{A}$.

For example: The inverse of a 2x2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is obtained as

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det \mathbf{A}} \quad \text{---(a)}$$

we have $\det \mathbf{A} = a_{11}a_{22} - a_{21}a_{12}$ ---(i)

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}; \text{ where, } C_{ij} = (-1)^{i+j} \det M_{ij}$$

$$\text{Therefore, } C_{11} = (-1)^{1+1} \det M_{11} = (-1)^2 a_{22} = a_{22} \quad \because \det M_{11} = |a_{22}| = a_{22}$$

$$C_{12} = (-1)^{1+2} \det M_{12} = (-1)^3 a_{21} = -a_{21}$$

$$C_{21} = (-1)^{2+1} \det M_{21} = (-1)^3 a_{12} = -a_{12}$$

$$C_{22} = (-1)^{2+2} \det M_{22} = (-1)^4 a_{11} = a_{11}$$

$$\therefore \mathbf{C} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} \text{ and hence, } \therefore \mathbf{C}^T = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \quad \text{---(ii)}$$

Substituting Eqns. (i) & (ii) into Eqn. (a), we get

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

5. **Partitioning of Matrices:** It is a process of subdividing a matrix of higher order into a number of submatrices of lower order. A submatrix is a matrix that is obtained from the original matrix by including only the elements of certain rows and columns.

For example, consider a set of 3 simultaneous equations expressed in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \quad \text{---(a)}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

Then, matrices $\mathbf{A}$, $\mathbf{X}$ and $\mathbf{b}$ can be partitioned or subdivided as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \text{ where } A_{11} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}; A_{12} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \text{ etc.}$$

$$\mathbf{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}; \text{ where } X_1 = \{x_1\} \text{ and } X_2 = \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} \text{ and}$$

$$\mathbf{b} = \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix}; \text{ where } B_1 = \{b_1\} \text{ and } B_2 = \begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}$$

Then, the complete set of Eqns. (a) can be written as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix}$$

In FEM, the concept of matrix partitioning is most useful to facilitate matrix manipulations and in reducing size of a system equations. It has advantages in saving computer storage as if any

submatrices repeat, it is necessary to store the submatrix only once. Using submatrices, we can identify a operation that repeated many times and we can carry out this operation only once and use the results whenever it is needed.

4. EIGEN VALUES AND EIGEN VECTOR PROBLEMS

The Eigen value problem is in the form of

$$Ay = \lambda y \quad \text{---(a)}$$

where A is a square matrix of dimension $(n \times n)$. It is required to find a non-zero Eigen vector ' y ' and the corresponding Eigen value ' λ ' that satisfy the above equation.

The above Eqn. (a) can be re-written as

$$[A - \lambda I]y = 0 \quad \text{---(b)}$$

The non-zero solution for ' y ' will occur when $[A - \lambda I]$ is a singular matrix.

$$\text{i.e. } \det [A - \lambda I] = 0 \quad \text{---(c)}$$

The Eqn. (c) is called characteristic equation and it can be solved for n roots which are called as Eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$.

For each Eigen value $\lambda_i (i = 1, 2, \dots, n)$ there is a Eigen vector y^i obtained from Eqn. (b).

The Eqn. (b) is expressed as

$$[A - \lambda_i I]y^i = 0$$

Problem 1: Calculate the Eigen values of the matrix $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$.

Solution.

The characteristic equation is

$$\det [A - \lambda I] = 0$$

$$\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1-\lambda & 2-0 \\ 2-0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$$

$$(-1-\lambda)(2-\lambda) - 2*2 = 0$$

$$-2 + \lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0 \text{ is the characteristic equation}$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4*1*-6}}{2} = \frac{1 \pm \sqrt{25}}{2} = -2, +3$$

i.e $\lambda_1 = -2$ and $\lambda_2 = 3$ are the Eigen values of matrix A .

Note: If Eigen vectors are also required to be determined, they can be determined as follows.

We have, $[A - \lambda_i I]y^i = 0$

$$\text{Or } \begin{bmatrix} -1-\lambda_i & 2 \\ 2 & 2-\lambda_i \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^i = 0 \quad \text{---(a)}$$

Substitute $\lambda_i = \lambda_1 = -2$ into Eqn. (a) we get

$$\begin{bmatrix} -1+2 & 2 \\ 2 & 2+2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^1 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} Y_1^1 \\ Y_2^1 \end{bmatrix} = 0$$

The first equation of the above equation yields

$$Y_1^1 + 2Y_2^1 = 0 \text{ or}$$

$$Y_1^1 = -2Y_2^1 \quad \text{---(b)}$$

Normalizing the above Eqn. (b) i.e. putting $Y_2^1 = 1$ then $Y_1^1 = -2$

Therefore, Eigen vectors corresponds to the Eigen value $\lambda_1 = -2$ is

$$\begin{Bmatrix} Y_1^1 \\ Y_2^1 \end{Bmatrix} = \begin{Bmatrix} -2 \\ +1 \end{Bmatrix}$$

Similarly substitute $\lambda_1 = \lambda_2 = 3$ into Eqn. (a) we get

$$\begin{bmatrix} -1-3 & 2 \\ 2 & 2-3 \end{bmatrix} \begin{bmatrix} Y_1^2 \\ Y_2^2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} Y_1^2 \\ Y_2^2 \end{bmatrix} = 0$$

The first equation of the above equation yields

$$-4Y_1^2 + 2Y_2^2 = 0 \text{ or}$$

$$Y_1^2 = 1/2 Y_2^2 \quad \text{---(c)}$$

Normalizing the above Eqn. (c) i.e. putting $Y_2^2 = 1$ then $Y_1^2 = 0.5$

Therefore, Eigen vectors corresponds to the Eigen value $\lambda_2 = 3$ is

$$\begin{Bmatrix} Y_1^2 \\ Y_2^2 \end{Bmatrix} = \begin{Bmatrix} +0.5 \\ +1 \end{Bmatrix}$$

Problem 2: Calculate Eigen values and vectors of matrix $A = \begin{bmatrix} 4 & -2.236 \\ -2.236 & 8 \end{bmatrix}$

Positive Definite Matrix: A symmetric matrix is said to be positive definite matrix if all its Eigen values are strictly positive (greater than zero). In the previous example problem-1, the matrix had $\lambda_1 = -2$ & $\lambda_2 = +3$ and hence it was not a positive definite matrix. On the other hand, in example problem-2, all Eigen values ($\lambda_1 = 3$ & $\lambda_2 = 9$) are positive and hence it is a positive definite matrix.

6. METHOD OF SOLUTION OF LINEAR ALGEBRAIC EQUATIONS

a. Gauss Elimination Method: Gauss elimination method is a well-known method of solving simultaneous equations by successively eliminating unknowns. For examples, if there are 'n' number of simultaneous equations to find 'n' number of unknowns $x_1, x_2, \dots, x_n$ then gauss elimination method successively eliminates variables $x_1, x_2, \dots, x_{n-1}$ until only one variable x_n is left. This results in an upper triangular matrix with reduced coefficients and reduced right side. The unknown variables $x_n, x_{n-1}, \dots, x_2, x_1$ are then successively determined by the process of back-substitution. The general procedure and algorithm of this method can be understood by considering the following simultaneous equations.

$$\left. \begin{array}{ll} x_1 - 2x_2 + 6x_3 = 0 & \text{---I} \\ 2x_1 + 2x_2 + 3x_3 = 3 & \text{---II} \\ -x_1 + 3x_2 = 2 & \text{---III} \end{array} \right\} \quad \text{---(A)}$$

The above equation can be solved, by Gauss elimination technique, using the following systematic steps:

Step-1, Eliminate x_1 from Eqns II and III in Eqn (A): Subtract a multiple of $\frac{2}{1}$ times the Eqn. I

from Eqn. II and a multiple of $\frac{-1}{1}$ times the Eqn. I from Eqn. III. The resulting equations which are modified once are as given in eqn. (B). 16

$$\left. \begin{array}{l} x_1 - 2x_2 + 6x_3 = 0 \\ 0 + 6x_2 - 9x_3 = 3 \\ 0 + x_2 + 6x_3 = 2 \end{array} \right\} \begin{array}{l} \text{---I} \\ \text{---II}^{(1)} \\ \text{---III}^{(1)} \end{array} \quad \text{---(B)}$$

Step-2, Eliminate x_2 from Eqn. III in eqn. (B): Subtract a multiple of $\frac{1}{6}$ times the Eqn. II⁽¹⁾ from Eqn. III⁽¹⁾. The resulting equations are as given in Eqn. (C).

$$\left. \begin{array}{l} x_1 - 2x_2 + 6x_3 = 0 \\ 0 + 6x_2 - 9x_3 = 3 \\ 0 + 0 + \frac{15}{2}x_3 = \frac{3}{2} \end{array} \right\} \begin{array}{l} \text{---I} \\ \text{---II}^{(1)} \\ \text{---III}^{(2)} \end{array} \quad \text{---(C)}$$

The variable x_1 has been eliminated from Eqns. II and III after modifying these equations once and x_2 has been eliminated from Eqn. III after modifying this equation twice. The superscript (1) and (2) in the above equations denotes the fact that the equations have been modified once and twice respectively.

Following the procedure explained in steps 1 and 2, a general algorithm for the reduction scheme of any variable at k^{th} step can be written as

Step-k:

$$\begin{aligned} a_{ij}^k &= a_{ij}^{k-1} - \frac{a_{ik}^{k-1}}{a_{kk}^{k-1}} a_{kj}^{k-1}; & i, j &= (k+1), \dots, n \\ b_i^k &= b_i^{k-1} - \frac{a_{ik}^{k-1}}{a_{kk}^{k-1}} b_k^{k-1}; & i &= (k+1), \dots, n \end{aligned} \quad \text{---(D)}$$

After $(n-1)$ steps, the 'n' number of simultaneous equations reduces to the form

$$\left[\begin{array}{cccc} a_{11} & a_{12} & - & - \\ & a_{22}^{(1)} & a_{23}^{(1)} & - \\ & & a_{33}^{(2)} & - \\ & & & a_{nn}^{(n-1)} \end{array} \right] \left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{array} \right\} = \left\{ \begin{array}{c} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{array} \right\}$$

Then the unknown variable x_n is given by

$$x_n = \frac{b_n}{a_{nn}}$$

The remaining unknown variables $x_{n-1}, x_{n-2}, \dots, x_2, x_1$ are obtained by the back substitution process and is given by

$$x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij} x_j}{a_{ii}}; \quad i = n-1, n-2, \dots, 1$$

In the above equations the superscripts are dropped for the convenience.

Prob. 1: solve the following system of simultaneous equations by Gauss elimination method.

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 7 \\ x_1 - 2x_2 + 3x_3 &= 12 \\ 2x_1 + 4x_2 - 3x_3 &= 10 \end{aligned}$$

Solution: The above simultaneous equations can be written in matrix form as

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 4 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 12 \\ 10 \end{Bmatrix}$$

At step k, the general algorithm for reduction scheme of unknown variables x_1, x_2 can be written as

$$a_{ij}^k = a_{ij}^{k-1} - \frac{a_{ik}^{k-1}}{a_{kk}^{k-1}} a_{kj}^{k-1}; \quad i, j = (k+1), \dots, n$$

$$b_i^k = b_i^{k-1} - \frac{a_{ik}^{k-1}}{a_{kk}^{k-1}} b_k^{k-1}; \quad i = (k+1), \dots, n$$

Step-1: Here $k=1$ and hence $i, j = (k+1), \dots, n=2, 3$ because, $k=1$ and $n=3$

(a) Modification of Eqn. II: $i = 2, j = 2, 3$

$$a_{22}^1 = a_{22}^0 - \frac{a_{21}^0}{a_{11}^0} a_{12}^0 = (-2) - \left(\frac{1}{2}\right) * 1 = -\frac{5}{2}$$

$$a_{23}^1 = a_{23}^0 - \frac{a_{21}^0}{a_{11}^0} a_{13}^0 = 3 - \left(\frac{1}{2}\right) * (-1) = \frac{7}{2}$$

$$b_2^1 = b_2^0 - \frac{a_{21}^0}{a_{11}^0} b_1^0 = 12 - \left(\frac{1}{2}\right) * 7 = \frac{17}{2}$$

The modified Eqn. II becomes as

$$0x_1 - \frac{5}{2}x_2 + \frac{7}{2}x_3 = \frac{17}{2} \quad \text{---II}^{(1)}$$

(b) Modification of Eqn. III: $i = 3, j = 2, 3$

$$a_{32}^1 = a_{32}^0 - \frac{a_{31}^0}{a_{11}^0} a_{12}^0 = 4 - \left(\frac{2}{2}\right) * 1 = 3$$

$$a_{33}^1 = a_{33}^0 - \frac{a_{31}^0}{a_{11}^0} a_{13}^0 = -3 - \left(\frac{2}{2}\right) * (-1) = -2$$

$$b_3^1 = b_3^0 - \frac{a_{31}^0}{a_{11}^0} b_1^0 = 10 - \left(\frac{2}{2}\right) * 7 = 3$$

The modified Eqn. III becomes as

$$0x_1 + 3x_2 - 2x_3 = 3 \quad \text{---III}^{(1)}$$

after step-1, Eqns. I, II⁽¹⁾ and III⁽¹⁾ can be written in matrix form as

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -\frac{5}{2} & \frac{7}{2} \\ 0 & 3 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ \frac{17}{2} \\ 3 \end{Bmatrix} \quad \begin{array}{l} \text{--- I} \\ \text{--- II}^{(1)} \\ \text{--- III}^{(1)} \end{array}$$

Step-2: Here $k=2$ and hence $i, j = (2+1)=3$

(a) Modification of Eqn. III⁽¹⁾: $i = 3, j = 3$

$$a_{33}^2 = a_{33}^1 - \frac{a_{32}^1}{a_{22}^1} a_{23}^1 = -2 - \left(\frac{3}{-5/2}\right) * (7/2) = \frac{11}{5}$$

$$b_3^2 = b_3^1 - \frac{a_{32}^1}{a_{22}^1} b_2^1 = 3 - \left(\frac{3}{-5/2} \right) * (17/2) = 66/2$$

The modified Eqn. III⁽¹⁾ reduces to

$$0x_1 + 0x_2 + \frac{11}{5}x_3 = \frac{66}{2} \quad \text{---III}^{(2)}$$

After step-2, the simultaneous Eqns reduces to the form

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -5/2 & 2 \\ 0 & 0 & 11/5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 17 \\ 66/5 \end{Bmatrix} \quad \begin{array}{l} \text{--- I} \\ \text{--- II}^{(1)} \\ \text{--- III}^{(2)} \end{array}$$

Then unknown variable x_3 is given by

$$\frac{11}{5}x_3 = \frac{66}{5}$$

$$x_3 = \frac{66/5}{11/5} = 6 \quad \Rightarrow x_n = \frac{b_n}{a_{nn}}, \text{ here } n=3$$

The remaining variables x_2, x_1 are obtained from

$$x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij}x_j}{a_{ii}}; \quad i = n-1, n-2, \dots, 1$$

Here $i = 3-1, 3-2 = 2, 1$

For $i=2$

$$x_2 = \frac{b_2 - \sum_{j=3}^3 a_{2j}x_j}{a_{22}} = \frac{b_2 - a_{23}x_3}{a_{22}} = \frac{17/2 - (7/2)*6}{-5/2} = 5$$

For $i=1$

$$x_1 = \frac{b_1 - \sum_{j=2}^3 a_{1j}x_j}{a_{11}} = \frac{b_1 - (a_{12}x_2 + a_{13}x_3)}{a_{11}} = \frac{7 - (1*5 - 1*6)}{2} = 4$$

Prob.2: Solve the following simultaneous equation using Gauss elimination method

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad \text{Ans: After 3rd step} \quad \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & 14/5 & -16/5 & 1 \\ 0 & 0 & 15/7 & -20/7 \\ 0 & 0 & 0 & 5/6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 8/7 \\ 7/6 \end{Bmatrix}$$

b. Frontal Solution Method or Wave Front Solution Method: Frontal solution method is an efficient method for solving finite element equations. In this method, the assembly of the global system stiffness matrix and reduction of equations are performed together. That is instead of assembling the entire system matrix and eliminating the unknowns as done in Gauss elimination method, this method eliminates the d.o.f (unknowns) while the rows and columns of the stiffness matrix corresponding to that d.o.f are assembled. Thus, the back-up storage for the

entire unreduced stiffness matrix is not required and there will be a considerable reduction in computer memory requirements, especially for the systems having large number of elements. This method can be described with the help of one-dimensional bar elements shown in Fig.2. Let k_1, k_2, k_3 and k_4 be the stiffness of elements 1, 2, 3 and 4 respectively. As there are 5 nodes, let us begin with 5x5 null matrix [K].

Adding the element stiffness matrix for “element-1” to the system matrix (null matrix) yields,

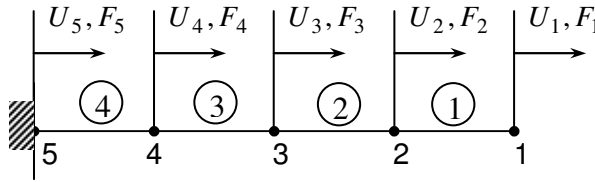


Fig.2: One- Dimensional Bar Elements

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} & = & \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} \end{matrix} \quad \text{---(A)} \quad \left| \quad K^1 = \begin{bmatrix} 1 & 2 \\ k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \right.$$

Since displacement U_1 is associated only with element-1, it does not appear in any other equations and can be eliminated now. It can be done as follows:

First row of Eqn.A is,

$$k_1 U_1 - k_1 U_2 = F_1$$

$$\therefore U_1 = \frac{F_1}{k_1} + U_2.$$

Second row of Eqn. A is

$$-k_1 U_1 + k_1 U_2 = F_2$$

To eliminate U_1 from second equation substitute for U_1 , we get

$$-k_1 \left(\frac{F_1}{k_1} + U_2 \right) + k_1 U_2 = F_2 \Rightarrow -F_1 - k_1 U_2 + k_1 U_2 = F_2$$

$$\text{Or } 0 \cdot U_1 + 0 \cdot U_2 = F_1 + F_2$$

Therefore, elimination of U_1 from the 2nd row yields

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} & = & \begin{Bmatrix} F_1 \\ F_1 + F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} \end{matrix} \quad \text{---(B)}$$

Next, if we process element – 2 and add its stiffness matrix terms to the appropriate locations, the system matrix becomes

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} \end{matrix} = \begin{bmatrix} F_1 \\ F_1 + F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} \quad \text{---(C)}$$

$$K^2 = \begin{bmatrix} & 2 & 3 \\ & k_2 & -k_2 \\ & -k_2 & k_2 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Displacement U_2 does not appear in any remaining equations and is now eliminated to obtain

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} \end{matrix} = \begin{bmatrix} F_1 \\ F_1 + F_2 \\ F_1 + F_2 + F_3 \\ F_4 \\ F_5 \end{bmatrix} \quad \text{---(D)}$$

Processing of the remaining elements in similar manner and following the elimination procedure results the final equation as

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ 0 & k_2 & -k_2 & 0 & 0 \\ 0 & 0 & k_3 & -k_3 & 0 \\ 0 & 0 & 0 & k_4 & -k_4 \\ 0 & 0 & 0 & -k_4 & k_4 \end{bmatrix} & \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} \end{matrix} = \begin{bmatrix} F_1 \\ F_1 + F_2 \\ F_1 + F_2 + F_3 \\ F_1 + F_2 + F_3 + F_4 \\ F_5 \end{bmatrix} \quad \text{---(E)}$$

As the last equation in Eqn. (E) corresponds to the boundary condition ($U_5 = 0$), it can be eliminated and remaining equations can be easily solved by back substitution. Note that, this method triangularized the system equation without assembling the entire stiffness matrix of the system.

6. NUMERICAL INTEGRATION

Gaussian Quadrature or Gauss – Legendre Numerical Integration Method:

In finite element method, the computation of element stiffness matrix or load vector requires integration of polynomial functions. A simple polynomial function is relatively easy to integrate. However, in many cases the ratios of polynomials are to be integrated and these are quite tedious to integrate directly. In such cases integration is carried out using numerical techniques. There are many numerical techniques and the most popular technique is Gaussian quadrature or Gauss-Legendre method.

For the illustration of Gaussian quadrature method, consider the one-dimensional integral of the form

$$I = \int_{-1}^{+1} f(r) dr \quad \text{---(a)}$$

Gaussian method approximate the above integration as

$$I = \sum_{i=1}^m W_i f(r_i) = W_1 f(r_1) + W_2 f(r_2) + \dots + W_m f(r_m) \quad \text{---(b)}$$

where, W_i and r_i are weights and sampling points. Use of m sampling points and weights results in an exact value of integrals for a polynomial of order $2m - 1$, if the sampling points and weights are chosen in accordance with Table 1 given below.

Table 1: Gauss points and weights for Gaussian quadrature integration

No. of points (m)	Locations (r_i)	Weights (W_i)
1	$r_1 = 0.0$	$W_1 = 2.0$
2	$r_{1,2} = \pm 0.5773502692 \quad (= \pm 1/\sqrt{3})$	$W_{1,2} = 1.0$
3	$r_{1,3} = \pm 0.7745966692$ $r_2 = 0.0$	$W_{1,3} = 0.5555555556$ $W_2 = 0.8888888889$
4	$r_{1,4} = \pm 0.8611363116$ $r_{2,3} = \pm 0.3399810436$	$W_{1,4} = 0.3478548451$ $W_{2,3} = 0.6521451549$

Note:

In Gaussian method, the integration limits should be between -1 to $+1$.

The number of sampling points are given by $m = \frac{n+1}{2}$.

The procedure can be extended to two- and three – dimension problems also. Therefore, the two – dimensional integral is expressed as

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(r_i, s_i) dr ds = \int_{-1}^{+1} \sum_{i=1}^m W_i f(r_i, s) ds = \sum_{j=1}^m \left[\sum_{i=1}^m W_i f(r_i, s_j) \right]$$

$$\text{or } I = \sum_{i=1}^m \sum_{j=1}^m W_i W_j f(r_i, s_j)$$

For two – point formula ($m=2$), above equation becomes as

$$I = W_1 W_1 f(r_1, s_1) + W_1 W_2 f(r_1, s_2) + W_2 W_1 f(r_2, s_1) + W_2 W_2 f(r_2, s_2)$$

Similarly, for three-dimension integration, the Gaussian formula is written as

$$I = \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m W_i W_j W_k f(r_i, s_j, t_k)$$

Problem 1: Evaluate the integral $I = \int_{-1}^{+1} (r^2 - 3r + 7) dr$ using Gaussian quadrature so that the result is exact.

Solution: The integrand $f(r) = r^2 - 3r + 7$ is a polynomial of order 2, therefore the number of sampling points and weights for exact integration are given by

$m = \frac{n+1}{2} = \frac{2+1}{2} = 1.5$. The calculated number of sampling points must be rounded up to nearest integer value. Hence in this case $m = 2$.

$$I = \sum_{i=1}^2 W_i f(r_i) = W_1 f(r_1) + W_2 f(r_2) \quad \text{---(a)}$$

For two point formula we have $r_1 = +0.5773502692$; $r_2 = -0.5773502692$; $W_1 = W_2 = 1.0$

$$f(r) = r^2 - 3r + 7$$

$$f(r_1) = 0.5773502692^2 - 3 * 0.5773502692 + 7 = 5.6012825257$$

$$f(r_2) = (-0.5773502692)^2 + 3 * 0.5773502692 + 7 = 9.0653841409$$

Substituting into Eqn. (a)

$$I = 1 * 5.6012825257 + 1 * 9.0653841409 = 14.66667$$

Note: The exact value of integral is

$$I = \int_{-1}^{+1} (r^2 - 3r + 7) dr = \left[\frac{r^3}{3} - \frac{3r^2}{2} + 7r \right]_{-1}^{+1} = \frac{2}{3} + 0 + 7 * 2 = 14.66667. \text{ Hence the two-point formula}$$

provides exact result in this case.

Problem 2: Evaluate the integral

$$I = \int_{-1}^{+1} \frac{r^2 - 1}{(r+3)^2} dr \text{ using Gauss-Legendre integration with one, two and three integration points.}$$

$$\text{Solution: Here function } f(r) = \frac{r^2 - 1}{(r+3)^2} \quad \text{---(a)}$$

a. One-point integration

For one point integration, the integration formula is

$$I = W_1 f(r_1)$$

where, $r_1 = 0.0$ and $W_1 = 2.0$.

$$I = W_1 f(r_1) = 2.0 * \left(\frac{0-1}{(0+3)^2} \right) = -0.2222222$$

b. Two-point integration

For two point integration, the integration formula is

$$I = \sum_{i=1}^2 W_i f(r_i) = W_1 f(r_1) + W_2 f(r_2) \quad \text{---(b)}$$

where, $r_1 = 1/\sqrt{3}$; $r_2 = -1/\sqrt{3}$. ; $W_1 = W_2 = 1.0$. Substitute $r = r_1$ and $r = r_2$ into Eqn. (a), we get

$$f(r_1) = \frac{(1/\sqrt{3})^2 - 1}{(1/\sqrt{3} + 3)^2} = -0.0520937 \text{ and } f(r_2) = \frac{(-1/\sqrt{3})^2 - 1}{(-1/\sqrt{3} + 3)^2} = -0.1135867$$

Substitution of $W_1, W_2, f(r_1), f(r_2)$ into Eqn. (b) yields

$$I = -0.1656804$$

c. Three-point integration

$$\text{For three point integration, } I = \sum_{i=1}^3 W_i f(r_i) = W_1 f(r_1) + W_2 f(r_2) + W_3 f(r_3) \quad \text{---(c)}$$

Where, $r_1 = 0.7745966692$; $r_2 = 0.0$; $r_3 = -0.7745966692$ and

$W_1 = W_3 = 0.5555555556; W_2 = 0.8888888889$. Then,

$$f(r_1) = \frac{0.7745966692^2 - 1}{(0.7745966692 + 3)^2} = \dots\dots\dots$$

$$f(r_2) = \frac{0 - 1}{(0 + 3)^2} = \dots\dots\dots$$

$$f(r_3) = \frac{(-0.7745966692)^2 - 1}{(-0.7745966692 + 3)^2} = \dots\dots\dots$$

Substitution of $W_1, W_2, W_3, f(r_1), f(r_2), f(r_3)$ into Eqn. (c) yields

$I = -0.15923$ (Please verify this answer)

Note: If we continue with four-point integration, we get $I = -0.15891$ (please check for the curiosity) and the exact result that obtained by direct integration is $I = -0.15888$. Thus, Gaussian integration does not always give exact solution but it provides converged solution as number of sampling points is increased.

Problem 3: Evaluate $I = \int_{-1}^{+1} \left(16x^3 + 6x^2 + \frac{1}{(x+1)} \right) dx$. Use two point Gauss quadrature.

(VTU, July 2004, M.Tech)

Solution: Here, $f(x) = \left(16x^3 + 6x^2 + \frac{1}{(x+1)} \right)$ ---(a)

The two-point integration formula is

$$I = W_1 f(x_1) + W_2 f(x_2) \quad \text{---(b)}$$

where, $x_{1,2} = \pm 1/\sqrt{3}$ and $W_1 = W_2 = 1.0$

Substituting $x = x_1$ and $x = x_2$ into Eqn. (a) we get

$$f(x_1) = \left(16 * \left(+\frac{1}{\sqrt{3}} \right)^3 + 6 * \left(\frac{1}{\sqrt{3}} \right)^2 + \frac{1}{(1/\sqrt{3} + 1)} \right) = 5.713176$$

$$f(x_2) = \left(16 * \left(-\frac{1}{\sqrt{3}} \right)^3 + 6 * \left(-\frac{1}{\sqrt{3}} \right)^2 + \frac{1}{(-1/\sqrt{3} + 1)} \right) = 1.286824$$

$I = 7.0$ Ans.

Problem 3: Evaluate $\int_{-1}^{+1} \left[3e^x + x^2 + \frac{1}{(x+2)} \right] dx$ using one and two-point gauss quadrature.

(VTU, July 2004, July 2005 B.E)

(Please refer Introduction to FE in Engg. by Chandrupatla for the solution, Ans=7.0 & 8.7857).

Problem 4: Evaluate I using two point Gaussian Quadrature technique.

$$I = \int_{-1}^1 N^T N d\xi, \text{ where } N = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \quad \text{VTU, June 2008)$$

$$\text{Solution: Here } f(\xi) = N^T N = \begin{bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{bmatrix} \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(1-\xi)^2 & \frac{1}{4}(1-\xi)(1+\xi) \\ \frac{1}{4}(1+\xi)(1-\xi) & \frac{1}{4}(1+\xi)^2 \end{bmatrix} \dots$$

(a)

The two-point integration formula is

$$I = W_1 f(\xi_1) + W_2 f(\xi_2) \quad \text{---(b)}$$

where, $\xi_{1,2} = \pm 1/\sqrt{3}$ and $W_1 = W_2 = 1.0$

Substituting $\xi = \xi_1$ and $\xi = \xi_2$ into Eqn. (a) we get

$$f(\xi_1) = \begin{bmatrix} \frac{1}{4} \left(1 - \frac{1}{\sqrt{3}}\right)^2 & \frac{1}{4} \left(1 - \frac{1}{\sqrt{3}}\right) \left(1 + \frac{1}{\sqrt{3}}\right) \\ \frac{1}{4} \left(1 + \frac{1}{\sqrt{3}}\right) \left(1 - \frac{1}{\sqrt{3}}\right) & \frac{1}{4} \left(1 + \frac{1}{\sqrt{3}}\right)^2 \end{bmatrix} = \begin{bmatrix} 0.044658 & 0.1667 \\ 0.1667 & 0.62201 \end{bmatrix}$$

$$f(\xi_2) = \begin{bmatrix} \frac{1}{4} \left(1 + \frac{1}{\sqrt{3}}\right)^2 & \frac{1}{4} \left(1 + \frac{1}{\sqrt{3}}\right) \left(1 - \frac{1}{\sqrt{3}}\right) \\ \frac{1}{4} \left(1 - \frac{1}{\sqrt{3}}\right) \left(1 + \frac{1}{\sqrt{3}}\right) & \frac{1}{4} \left(1 - \frac{1}{\sqrt{3}}\right)^2 \end{bmatrix} = \begin{bmatrix} 0.62201 & 0.1667 \\ 0.1667 & 0.044658 \end{bmatrix}$$

Substitute into Eqn. (b) and carryout matrix addition, we get

$$I = 1 * \begin{bmatrix} 0.044658 & 0.1667 \\ 0.1667 & 0.62201 \end{bmatrix} + 1 * \begin{bmatrix} 0.62201 & 0.1667 \\ 0.1667 & 0.044658 \end{bmatrix} = \begin{bmatrix} 0.666668 & 0.3334 \\ 0.3334 & 0.666668 \end{bmatrix}$$

Problem 5: Evaluate $I = \int_1^3 \frac{dx}{x}$ by 3-point formula (VTU, July 2005, B.E)

Solution: Gaussian integration requires the integration limits from -1 to $+1$. Therefore, the argument of the above integral must be changed to represent integration limits -1 to $+1$ as follows.

Let a new argument be

$$r = Ax + B \quad \text{---(a)}$$

$$\therefore dr = Adx \quad \text{---(b)}$$

Where, the coefficients A and B are determined so that the integration limits becomes -1 to $+1$ as follows.

$$\text{When } x = 1, r = -1. \therefore -1 = A * 1 + B \quad \text{---(i)}$$

$$\text{When } x = 3, r = +1. \therefore +1 = A * 3 + B \quad \text{---(ii)}$$

(i) – (ii) $-2 = -2A$ or $A = 1$. Substitute for A in Eqn. (i) we get, $-1 = 1 + B$ or $B = -2$. Substitution of A and B into Eqn. (a) yields

$$r = 1 * x - 2 \text{ or } x = r + 2. \text{ Substitution of } A \text{ into Eqn. (b) yields } dx = 1 * dr.$$

Therefore, the given integration argument can be modified as

$$I = \int_{-1}^{+1} \frac{dr}{(r + 2)}$$

For 3-point integration we have

$$I = W_1 f(r_1) + W_2 f(r_2) + W_3 f(r_3) \quad \text{---(c)}$$

where, $r_{1,3} = \pm 0.7745966692$; $r_2 = 0.0$ and $W_{1,3} = 0.5555555556$; $W_2 = 0.8888888889$

$$\text{Here, } f(r) = \frac{1}{(r + 2)}$$

Substitution of r_1, r_2, r_3 into the above equation yields

$$f(r_1) = 0.3604127; f(r_2) = 0.5; f(r_3) = 0.810579.$$

Substitution of $F(r_i)$ and W_i into Eqn. (c) yields

$$I = 0.5555555556 * 0.3604127 + 0.8888888889 * 0.5 + 0.5555555556 * 0.810579$$

$$I = 1.098039 \text{ Ans.}$$

Problem 6: Evaluate the following integrals using two-point formula.

$$(a) \int_0^3 (x^2 - 1)dx \text{ and } (b) \int_1^6 (y^3 + 2y)dy$$

Solution:

$$(a) \int_0^3 (x^2 - 1)dx$$

$$\text{Let } r = Ax + B; \therefore dr = A dx$$

$$\therefore dr = A dx$$

$$\text{When, } x = 0, r = -1 \therefore -1 = A * 0 + B \text{ or } B = -1$$

$$\text{When, } x = 3, r = +1, \therefore +1 = A * 3 - 1 \text{ since, } B = -1 \text{ or } A = 2/3$$

$$\text{Therefore, } r = \frac{2}{3}x - 1 \text{ or } x = \frac{3}{2}(r + 1) \text{ and } dr = A dx = \frac{2}{3} dx \text{ or } dx = \frac{3}{2} dr$$

$$\text{Therefore, } (x^2 - 1)dx = \left(\frac{9}{4}(r + 1)^2 - 1 \right) * \frac{3}{4} dr = \left(\frac{27}{8}(r + 1)^2 - \frac{3}{2} \right) dr$$

$$\text{Hence, } I = \int_{-1}^{+1} \left(\frac{27}{8}(r + 1)^2 - \frac{3}{2} \right) dr$$

$$\text{Proceed same as previous problems to get } f(r_1) = 6.897114; f(r_2) = -0.8971143 \text{ and } I = 6.0$$

Ans.

$$(b) \int_1^6 (y^3 + 2y)dy$$

$$\text{Proceed same as previous problem and show that } y = \left(\frac{5}{2}r + \frac{7}{2} \right) \& dy = \frac{5}{2} dr.$$

Change the given integration argument using the above relation and proceed.

Two-Dimensional Integration:

$$\text{Problem 7: Evaluate } I = \int_{-1}^{+1} \int_{-1}^{+1} (r^3 - 1)(s - 1)^2 dr ds \text{ using two-point integration.}$$

$$\text{Solution: The function is } f(r, s) = (r^3 - 1)(s - 1)^2$$

The integration formula for 2-D problem with two-point integration is

$$I = \sum_{i=1}^2 \sum_{j=1}^2 W_i W_j f(r_i, s_j) = W_1 W_1 f(r_1, s_1) + W_1 W_2 f(r_1, s_2) + W_2 W_1 f(r_2, s_1) + W_2 W_2 f(r_2, s_2)$$

---(b)

$$\text{where, } r_1 = s_1 = 1/\sqrt{3}; r_2 = s_2 = -1/\sqrt{3} \text{ and } W_1 = W_2 = 1.0$$

$$f(r_1, s_1) = (r_1^3 - 1)(s_1 - 1)^2 = \left((1/\sqrt{3})^3 - 1 \right) (1/\sqrt{3} - 1)^2 = -0.1442549$$

$$f(r_1, s_2) = (r_1^3 - 1)(s_2 - 1)^2 = \left((1/\sqrt{3})^3 - 1 \right) (-1/\sqrt{3} - 1)^2 = -2.0092115$$

$$f(r_2, s_1) = (r_2^3 - 1)(s_1 - 1)^2 = \left((-1/\sqrt{3})^3 - 1 \right) (1/\sqrt{3} - 1)^2 = -0.2130107$$

$$f(r_2, s_2) = (r_2^3 - 1)(s_2 - 1)^2 = \left((-1/\sqrt{3})^3 - 1 \right) (-1/\sqrt{3} - 1)^2 = -2.96685578$$

Substituting into Eqn. (b) we get

$$I = 1 * 1 * (-0.1442549) + 1 * 1 * (-2.0092115) + 1 * 1 * (-0.2130107) + 1 * 1 * (-2.96685578)$$

$$I = -5.333333 \text{ Ans.}$$

Problem 8: Using Gaussian quadrature, evaluate the following integral by two point formula (VTU, July 2003, B.E) 10 Marks

$$\int_{-1}^{+1} \int_{-1}^{+1} (\xi^2 + 2\eta\xi + \eta^3) d\xi d\eta$$

7. MATRIX NOTATION

The finite element formulations are expressed in matrix form, because it is only in matrix form that the complete solution process can be expressed in a compact and elegant manner. For a 1 D structural problem, as an example, which is discretized to have n number of nodes, the equilibrium equation that relates the nodal displacement with applied load is expressed as

$$\begin{bmatrix} K_{11} & K_{12} & \dots & K_{n1} \\ K_{21} & K_{22} & \dots & K_{n2} \\ | & | & & | \\ | & | & & | \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix}_{(n \times n)} \begin{Bmatrix} u_1 \\ u_2 \\ | \\ | \\ u_n \end{Bmatrix}_{(n \times 1)} = \begin{Bmatrix} F_1 \\ F_2 \\ | \\ | \\ F_n \end{Bmatrix}_{(n \times 1)}$$

In short form $[K]\{q\} = \{P\}$

Where $[K]$ = Stiffness matrix ($n \times n$ square matrix) and $\{q\}$ and $\{P\} = n \times 1$ column vector of unknowns (displacements) and loads respectively.

In general, the two-dimensional matrix such as $[K]$ is denoted by square brackets, $[]$, individual elements of such matrix (K_{11}, K_{12} , etc.) are represented by doubly subscripted letters. First subscript gives row number and second gives the column number. The column vectors such as $\{q\}$ and $\{P\}$ are denoted by braces, $\{ \}$ and its elements by singly subscripted letters. Row vector is indicated as the transpose of a column vector or vice versa.

i.e. $\{q\} = \{u_1 \ u_2 \ \dots \ u_n\}^T$ or $\{q\}^T = \{u_1 \ u_2 \ \dots \ u_n\}$

The row vector is denoted by $[]$. Using this notation the following expression can be written as

$$u = N_1 u_1 + N_2 u_2 + \dots + N_n u_n = [N_1 \ N_2 \ \dots \ N_n] \begin{Bmatrix} u_1 \\ u_2 \\ | \\ u_n \end{Bmatrix}$$

The above equation can be written in short form as

$u = [N]\{q\}$, where $[N]$ is a $1 \times n$ row vector and its transpose is

$$[N]^T = \begin{Bmatrix} N_1 \\ N_2 \\ | \\ N_n \end{Bmatrix}$$

Example 1: If $\pi = \frac{1}{2} \{q\}^T [K] \{q\} - \{q\}^T \{F\}$, where $\{q\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$; $[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ and 27

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}, \text{ show that } \pi = \frac{1}{2} (K_{11}u_1^2 + K_{21}u_1u_2 + K_{12}u_1u_2 + K_{22}u_2^2) - (F_1u_1 + F_2u_2)$$

$$\textbf{Proof: } [K]\{q\} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_{2 \times 2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{2 \times 1} = \begin{Bmatrix} K_{11}u_1 + K_{12}u_2 \\ K_{21}u_1 + K_{22}u_2 \end{Bmatrix}$$

$$\text{and } \{q\}^T \{F\} = \begin{Bmatrix} u_1 & u_2 \end{Bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = (F_1u_1 + F_2u_2)$$

$$\{q\}^T [K]\{q\} = \begin{Bmatrix} u_1 & u_2 \end{Bmatrix}_{1 \times 2} \begin{bmatrix} K_{11}u_1 + K_{12}u_2 \\ K_{21}u_1 + K_{22}u_2 \end{bmatrix}_{2 \times 1} = \{(K_{11}u_1 + K_{12}u_2)u_1 + (K_{21}u_1 + K_{22}u_2)u_2\}_{1 \times 1}$$

$$\therefore \pi = \frac{1}{2} \{q\}^T [K]\{q\} - \{q\}^T \{F\} = \frac{1}{2} (K_{11}u_1^2 + K_{12}u_1u_2 + K_{21}u_1u_2 + K_{22}u_2^2) - (F_1u_1 + F_2u_2)$$

Example 2: If $\pi = \frac{1}{2} \{q\}^T [K]\{q\} - \{q\}^T \{F\}$ and matrices $\{q\}$, $[K]$ and $\{F\}$ are as defined in the example 1, show that $\left\{ \frac{\partial \pi}{\partial q_i} \right\} = [K]\{q\} - \{F\}$ for $i = 1, 2$. Matrix $[K]$ is square and symmetric matrix.

Proof: From example 1, we have

$$\pi = \frac{1}{2} \{q\}^T [K]\{q\} - \{q\}^T \{F\} = \frac{1}{2} (K_{11}u_1^2 + K_{12}u_1u_2 + K_{21}u_1u_2 + K_{22}u_2^2) - (F_1u_1 + F_2u_2)$$

Differentiate w.r.t u_1 , we get

$$\frac{\partial \pi}{\partial u_1} = \frac{1}{2} (2K_{11}u_1 + K_{12}u_2 + K_{21}u_2) - F_1. \text{ For symmetric matrix } K_{12} = K_{21}, \text{ hence above equation}$$

reduces to

$$\frac{\partial \pi}{\partial u_1} = \frac{1}{2} (2K_{11}u_1 + 2K_{12}u_2) - F_1 = (K_{11}u_1 + K_{12}u_2) - F_1 \quad \text{---(a)}$$

$$\text{Similarly, } \frac{\partial \pi}{\partial u_2} = (K_{21}u_1 + K_{22}u_2) - F_2 \quad \text{---(b)}$$

Eqns. (a) and (b) can be written in matrix form as

$$\begin{Bmatrix} \frac{\partial \pi}{\partial u_1} \\ \frac{\partial \pi}{\partial u_2} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = [K]\{q\} - \{F\} \quad \therefore \left\{ \frac{\partial \pi}{\partial q_i} \right\} = \begin{Bmatrix} \frac{\partial \pi}{\partial u_1} \\ \frac{\partial \pi}{\partial u_2} \end{Bmatrix} = [K]\{q\} - \{F\}$$